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BOUNDS TO THE INFLUENCE COEFFICIENTS BY THE ASSUMED STRESS METHOD†

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Abstract—The direct influence coefficient obtained by the hybrid stress finite element model proposed by Pian has been proved to be bounded from above from that of the equilibrium model, provided that they have the same type of stress distribution within each element, and that of the compatible displacement model, provided that they are the same type of displacement along the interelement boundary.

IN APPLYING the finite element methods in structural mechanics Fraeijs de Veubeke [1] has shown that the direct influence coefficients are bounded from above by that derived from the equilibrium model and bounded from below by that derived from the compatible displacement model. In a previous paper [2] the authors have pointed out that, under certain restrictions, this is also true for the direct influence coefficients derived by the assumed stress hybrid method [3, 4]. In this note we shall give a detailed derivation of the bounds of the direct influence coefficients.

Let us consider the deformation of a plate. For simplicity we shall restrict ourselves to the case of homogeneous boundary conditions only. The three variational functionals of the plate for the compatible displacement model, the assumed stress hybrid model and the equilibrium models are, respectively,

$$\begin{aligned}\Pi_d &= \sum_m \Pi_{md}\\ \Pi_h &= \sum_m \Pi_{mh}\\ \Pi_e &= \sum_m \Pi_{me}\end{aligned} \tag{1}$$

where m indicates the mth element, Σ_m sums over all elements and

$$\Pi_{md} = \tfrac{1}{2}\int_{A_m} K_{\alpha\beta\lambda\theta} w_{,\alpha\beta} w_{,\lambda\theta}\,\mathrm{d}A - \int_{A_m} pw\,\mathrm{d}A \tag{2a}$$

$$\Pi_{mh} = \int_{\partial A_m} (M_{\alpha\beta} w_{,\alpha} v_\beta - M_{\alpha\beta,\beta} v_\alpha w)\,\mathrm{d}s - \tfrac{1}{2}\int_{A_m} C_{\alpha\beta\lambda\theta} M_{\alpha\beta} M_{\lambda\theta}\,\mathrm{d}A \tag{2b}$$

$$\Pi_{me} = -\tfrac{1}{2}\int_{A_m} C_{\alpha\beta\lambda\theta} M_{\alpha\beta} M_{\lambda\theta}\,\mathrm{d}A. \tag{2c}$$

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In the equations above the summation convention has been used for the repeated Greek subscripts. A_m is the area of an element and ∂A_m is the boundary of A_m, p is the external load, w is the outplane deflection, M's are the components of the moments, K's and C's are, respectively, the elastic stiffness constants and the flexibility constants. The admissibility conditions for the functionals in equation (1) are

(a) In Π_d both w and its first partial derivatives are continuous functions over the entire domain and satisfy the displacement boundary conditions.
(b) In Π_h both w and $w_{,\alpha}$, which satisfy the displacement boundary conditions, are defined along the interelement boundaries only, while $M_{\alpha\beta}$ are defined within each element and satisfy the equilibrium equations.
(c) In $\Pi_e, M_{\alpha\beta}$ satisfy the equilibrium equations within each element and across the interelement boundaries.

We shall first use Π_h and Π_e to establish an upper bound for the direct influence coefficient of the assumed stress hybrid method. Since

$$C_{\alpha\beta\lambda\theta}M_{\alpha\beta}M_{\lambda\theta} \geq 0$$

Π_h is a maximum principle with respect to the moment distribution M's, when w and $w_{,\alpha}$ along the interelement boundaries are fixed. In particular, if M's for each element are also in equilibrium with all their neighboring elements, we have

$$\sum_m \int_{\partial A_m} (M_{\alpha\beta}w_{,\alpha}\nu_\beta - M_{\alpha\beta,\beta}w\nu_\alpha)\,\mathrm{d}s = 0$$

i.e. in this case

$$\Pi_h = \Pi_e. \tag{3}$$

In the construction of the finite element equations [3, 4] we have M's expressed in terms of unknown β's, w and $w_{,\alpha}$ expressed in terms of unknown q's. If we write Π_h in a matrix form, namely [3]

$$\Pi_h = \sum_m (\boldsymbol{\beta}^T\mathbf{G}\mathbf{q} - \tfrac{1}{2}\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta} - \beta^T\mathbf{H}_p + q^TS + C). \tag{4}$$

(See equation (12) of Ref. [4] or equation (17) of Ref. 2) where $\mathbf{H}_p$, S and C depend only on external load. The maximum of Π_h with respect to all β's with fixed q's can be written as

$$\Pi_h(\mathbf{q}) = \sum_m \Pi_{mh}(\mathbf{q}) = \sum_m (\tfrac{1}{2}\mathbf{q}^T\mathbf{k}\mathbf{q} - \mathbf{q}^T\mathbf{Q} + \mathbf{C}) \tag{5}$$

where

$$\mathbf{k} = \mathbf{G}^T\mathbf{H}^{-1}\mathbf{G}.$$

If we choose the similar type of moment distribution for Π_e by imposing the subsidiary conditions to β's such that all the moments are also in equilibrium across the interelement boundaries, Π_e can be written as

$$\Pi_e(\boldsymbol{\beta}) = -\sum_m (\tfrac{1}{2}\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta} + \beta^T\mathbf{H}_p) \tag{6}$$

By equation (3) and from the fact that $\Pi_h(\mathbf{q})$ is the maximum of Π_h for all β's with q's fixed, we have

$$\Pi_h(\mathbf{q}) \geq \Pi_e(\boldsymbol{\beta}). \tag{7}$$

The finite element solution of the assumed stress hybrid model corresponding to the minimum, say Π_h^*, of $\Pi_h(\mathbf{q})$ with respect to all q's, while the finite element solution of the equilibrium model corresponds to the maximum, say Π_e^* of $\Pi_e(\boldsymbol{\beta})$ with respect to all unconstrained β's. By equation (7) we still have

$$\Pi_h^* \geq \Pi_e^*. \tag{8}$$

Let F_1 be a prescribed generalized force, the direct influence coefficient, c_{11}, may be defined by the equation

$$\begin{aligned} \Pi_h^* &= -\tfrac{1}{2}(c_{11})_h F_1^2 \\ \Pi_e^* &= -\tfrac{1}{2}(c_{11})_e F_1^2. \end{aligned} \tag{9}$$

By equation (8) we conclude that

$$(c_{11})_e \geq (c_{11})_h. \tag{10}$$

We shall now use Π_d and Π_h to show that the direct influence coefficient of the assumed stress hybrid model is bounded from below. In equations (2a) and (2b), if w and $w_{,\alpha}$ are the same for both Π_{md} and Π_{mh} on ∂A_m, then

$$\Pi_{md} \geq \Pi_{mh}. \tag{11}$$

This is because for any given w and $w_{,\alpha}$ on ∂A_m, Π_{md} and Π_{mh} are the potential energy and the complementary energy of the element A_m. In the construction of the finite element equations, Π_{md} is also expressed in terms of the unknown q's of the displacement

$$\Pi_{md}(\mathbf{q}) = \tfrac{1}{2}\mathbf{q}^T\mathbf{k}_d\mathbf{q} - \mathbf{q}^T\mathbf{S}_d. \tag{12}$$

If the interelement displacements are the same for both Π_{md} and Π_{mh}, according to equation (11), we have

$$\Pi_{md}(\mathbf{q}) \geq \Pi_{mh}(\mathbf{q})$$

or

$$\Pi_d(\mathbf{q})\left[=\sum_m \Pi_{md}(\mathbf{q})\right] \geq \Pi_h(\mathbf{q})\left[=\sum_m \Pi_{mh}(\mathbf{q})\right] \tag{13}$$

for all q's. The finite element solution of the displacement model will correspond to the minimum, say Π_d^* of $\Pi_d(q)$ with respect to all q's. By equation (13) we have

$$\Pi_d^* \geq \Pi_h^*. \tag{14}$$

The direct influence coefficient for Π_d associated with the prescribed generalized load F_1 is defined by

$$\Pi_d^* = -\tfrac{1}{2}(c_{11})_d F_1^2. \tag{15}$$

Therefore we have

$$(c_{11})_h \geq (c_{11})_d. \tag{16}$$

Once the bounds for the direct influence coefficient are established, the derivation of bounds to the cross influence coefficients is a simple algebraic process [5]. Let us consider the case of having two generalized prescribed loads F_1 and F_2,

$$\Pi_i^* = -[\tfrac{1}{2}(c_{11})_i F_1^2 + (c_{12})_i F_1 F_2 + \tfrac{1}{2}(c_{22})_i F_2^2]$$

where $i = d, h$ and e. We have

$$(c_{11})_d \leq (c_{11})_h \leq (c_{11})_e$$
$$(c_{22})_d \leq (c_{22})_h \leq (c_{22})_e$$

and

$$\Pi_d^* \geq \Pi_h^* \geq \Pi_e^*$$

then

$$\begin{aligned} |(c_{12})_h-(c_{12})_d| &\geq \sqrt{\{[(c_{11})_e-(c_{11})_d][(c_{22})_e-(c_{22})_d]\}} \\ |(c_{12})_h-(c_{12})_e| &\geq \sqrt{\{[(c_{11})_e-(c_{11})_d][(c_{22})_e-(c_{22})_d]\}}. \end{aligned} \tag{17}$$

In conclusion, we have shown in equations (10) and (16) that the direct influence coefficient of the assumed stress hybrid model is bounded from above by that of the equilibrium model, provided they have the same type of stress distribution within each element, and is bounded from below by that of the displacement model, provided they have the same type of displacement along the interelement boundaries. These are clearly demonstrated in Fig. 3 of Ref. [2]. The variational principle Π_h used for the assumed stress hybrid model is neither a maximum nor a minimum principle. In equations (4) and (5), since H is positive definite, the value of $\Pi_{mh}(\mathbf{q})$ will tend to increase if we increase the number of β's, while keeping the number of $\mathbf{q}$'s fixed. This, in turn, increases the value of $\Pi_{mh}(\mathbf{q})$ [or decreases the value of $(c_{11})_h$]. In equation (5), $\mathbf{k}$ is positive semi-definite, the value of $\Pi_{mh}(\mathbf{q})$ will tend to decrease if we increase the number of q's. So it has the effect of increasing $(c_{11})_h$. These are demonstrated in Fig. 5 of Ref. [2].

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Абстракт—Доказывается, что непосредственный козффициент влияния, полученный из модели смешанного конечного злемента для напряжении, предложенного Пцаном, является ограниченным сверху от такого же для модели равновесия, если только они имеют такой же самый тип распределения напряжении внутри каждого злемента. Зтот коеффициент ограничен также от такого же дла модели совместного перемещения, если они обладают таким же самым типом леремещении вдоль границы между злементами.